

SYMMETRY AND COMPLETE REGULARITY: KOPPERMAN'S DUALITY À LA QUANTALE.

JORGE BRUNO

ABSTRACT. Nearly three decades from his celebrated result, we study a modern refinement and strengthening of Kopperman's full metrisability of all topological spaces. Within this new theory of *V-spaces*, developed by Flagg and Weiss, we investigate several topological notions and their metric dual. Among our main results is the reconstruction, in terms of V-spaces, of Kopperman's equivalence between symmetric value semigroups and completely regular topologies. We conclude our work by revisiting some classical topological results and their almost evident validity through this metric lens.

1. INTRODUCTION

Let **Met** denote the category metric spaces with ϵ - δ continuous functions. The coincidence of both types of morphisms implies that the obvious functor **Met** \rightarrow **Top**, that sends any metric space to the topological space it generates, is fully faithful on **Top_M** - the category of metrisable topological spaces. The fullness of this functor is largely due to the triangle inequality imposed on any metric space. Were it not the case, it would only be true that ϵ - δ continuity implies topological continuity. Indeed, take $X = \{a, b, c, d\}$ with $d : X^2 \rightarrow \mathbb{R}$ as

$$d(x, y) = \begin{cases} 0 & \text{if } \{x, y\} = \{a, b\} \text{ or } \{x, y\} = \{b, c\}, \\ 2 & \text{if } \{x, y\} = \{a, c\}, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

The reader can quickly verify that $\text{int}[B_2(a)] = \{d\}$. It thus follows that ϵ - δ continuous functions are topologically continuous but the converse is certainly not true. Back to **Met** \rightarrow **Top**, it is then most natural to ask for a fully faithful extension $\mathcal{O} : \mathbf{C} \rightarrow \mathbf{Top}$ of **Met** \rightarrow **Top** so that the diagram

$$\begin{array}{ccc} \mathbf{Top} & \begin{array}{c} \xleftarrow{\mathcal{O}} \\ \xrightarrow{M} \end{array} & \mathbf{C} \\ \uparrow & & \uparrow \\ \mathbf{Top}_M & \begin{array}{c} \xleftarrow{\mathcal{O}} \\ \xrightarrow{M} \end{array} & \mathbf{Met} \end{array}$$

1

commutes and where the pair of functors (\mathcal{O}, M) represent an equivalence of categories. That is, a comprehensive and cohesive *metrisation* of all topological spaces where topological notions can be naturally interpreted by their metric counterparts.

The first attempt at creating this extension was achieved by Kopperman in terms of his *continuity spaces* - sets valued on value semi-groups. Kopperman's theory captures many of the properties from $([0, \infty], \leq, +)$ that make metric techniques more powerful than topological ones. A striking fact of this metrisation is the duality, illustrated by Kopperman in [6], between *symmetric* continuity spaces and their completely regular counterparts. A bold result that further illustrates the naturality of this metrisation; one would be inclined to expect such an equivalence given the well-known duality between completely regular topologies and uniform spaces.

In Kopperman's theory the concept of positive elements, naturally occurring in $([0, \infty], \leq, +)$, is not an intrinsic one. A refinement of Kopperman's approach is the one initiated by Flagg in [5] based on his quantale valued *V-spaces*. Given a value quantale V , a V -space is a pair (X, d) where X is any set and $d : X \times X \rightarrow V$ is a distance assignment of points from X valued in V so that:

- $d(x, x) = 0$ for all $x \in X$, and
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

In [8] Flagg's V -spaces are then captured by Weiss as objects within a suitable category, which he denotes by \mathbf{M}_T . Inspired on the duality between ϵ - δ continuity and its topological form, Weiss encompassed topological continuity in terms of a binary relation naturally present in value distributive lattices: the *well-above relation*. Within the last few years this latter metrisation has received large amounts of attention: in [9] a metric characterisation of topological connectedness is constructed; Ackerman extends Banach's fixed point theorem for a large class of symmetric V -spaces in [1]; the authors of [4] describe a generalisation of the classical completion of a metric space in terms of V -spaces and the concept of *uniformly vanishing asymmetry*; while a categorical framework for topological invariants is developed in [10].

In light of the above work, the aim of this paper is to further augment the study of Flagg's metrisation. More precisely, we are concerned with reconstructing some classical topological notions in \mathbf{M}_T . We begin by interpreting basic topological constructs (e.g., product spaces, box topologies, quotients, topological sums, etc) in \mathbf{M}_T and illustrating their complexity within this metric context. We then turn our attention to our main result: reproducing Kopperman's duality in terms of Flagg's V -spaces. That is, constructing an equivalence between symmetric V -spaces and completely regular topologies and thus

further evidence Flagg's metric approach to topology as the correct refinement of Kopperman's. With the above results at our disposal, we conclude our work by revisiting some classical topological results and their almost evident validity when interpreted through this metric lens. In particular, the latter assertion being a consequence of the intrinsic complexity of topological (co)limits in $\mathbf{M_T}$.

2. BACKGROUND

The construction of objects from $\mathbf{M_T}$ can be found in [5] (pg. 268) while the equivalence is illustrated in [8]. For completion, we summarise this equivalence here. For a lattice L and any pair $x, y \in L$, $y \succ x$ is the **well-above relation** defined by: $y \succ x$ if whenever $x \geq \bigwedge S$, with $S \subseteq L$, there exists some $s \in S$ such that $y \geq s$. The top and bottom elements of any lattice L will be denoted by \top_L and \perp_L , respectively. So as to lighten the notational burden, when we find ourselves in the presence of an indexed family of lattices L_i , their respective top and bottom elements' subscripts will be the lattice's index instead. That is \perp_i and \top_i rather than \perp_{L_i} and \top_{L_i} , respectively. A well-known characterization of completely distributive lattices is the following.

Theorem 1 (Raney [7]). *A lattice L is completely distributive if, and only if, for all $y \in L$*

$$y = \bigwedge \{a \in L \mid a \succ y\}.$$

Definition 2. *A value distributive lattice is a completely distributive lattice L for which $L_{\prec} = \{a \in L \mid a \succ 0\}$ forms a filter. A **value quantale** is a value distributive lattice L together with an associative and commutative binary operation $+$: $L \times L \rightarrow L$ such that for all $x \in L$*

- $x + \perp_L = x$, and
- $\bigwedge(x + S) = x + \bigwedge S$ for all $S \subseteq L$.

Value quantales will often be denoted with the letters V and W in order to syntactically distinguish them from standard completely distributive lattices.

Definition 3. *Let V be a value quantale. A **V -space** is a pair (X, d) with X a set and $d : X \times X \rightarrow V$ such that*

- $d(x, x) = 0$ for all $x \in X$, and
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Given any value quantale V and a V -space (X, d) we follow Flagg's adoption of Kopperman's terminology and denote the triple (V, X, d) as a **continuity space**. The category $\mathbf{M_T}$ will be that of all continuity spaces with a morphism $(V, X, d) \rightarrow (W, Y, m)$ being a function $f : X \rightarrow Y$ such that for every $x \in X$ and for every $\epsilon \in W_{\prec}$ there exists

$\delta \in V_{\prec}$ such that for all $x' \in X$: if $d(x, x') \prec \delta$ then $m(f(x), f(x')) \prec \epsilon$. We call these morphisms ϵ - δ **continuous functions**. One can readily verify that **Met** is a full subcategory of **M_T**: every ordinary metric space (X, d) is a V -space for $V = [0, \infty]$ with $+$ being ordinary addition.

Definition 4. Let (X, d) be a V -space and $\epsilon \in V$ with $\epsilon \succ 0$. $B_{\epsilon}(x) = \{y \in X \mid d(y, x) \prec \epsilon\}$ is the **open ball** with radius ϵ about the point $x \in X$.

Theorem 5 (Flagg). Let (X, d) be a V -space. Declaring a set $U \subseteq X$ to be open if for every $x \in U$ there exists $\epsilon \succ 0$ such that $B_{\epsilon}(x) \subseteq U$ defines a topology on X . Moreover, ϵ - δ continuity and topological continuity coincide.

Proof. The first part is proved in [5] as Theorem 4.2 and Theorem 4.4 while morphisms are dealt with in [8]. \square

The functor $\mathcal{O} : \mathbf{M}_T \rightarrow \mathbf{Top}$ will be the one that maps a continuity space to the topology it defines, as detailed in the previous result. In line with standard metrisable topologies, given a continuity space (V, X, d) we will refer to $\mathcal{O}[(V, X, d)]$ as the topology **generated** by (V, X, d) .

Before we describe \mathcal{O} 's inverse we introduce the following: for any collection of sets \mathcal{A} and $A \subseteq \mathcal{A}$, we say that A is **downwards closed** provided that $B, C \in \mathcal{A}$ and $B \subseteq C$, if $C \in A$ then $B \in A$. Also, we follow standard set-theoretic notation in that for any set X , we let $[X]^{<\omega}$ denote the collection of all finite subsets of X . Lastly, for any set X let

$$\Omega(X) = \{A \subseteq [X]^{<\omega} \mid A \text{ is downwards closed}\}.$$

Lemma 6 (Flagg). Given a set X , ordering $\Omega(X)$ by reverse set inclusion yields $(\Omega(X), \leq, +)$ as a value quantale where $+$ is given by intersection and $p \succ \perp$ if, and only if, p is finite.

Proof. This is part of Example 1.1 in [5]. \square

The following is also due to Flagg and we reproduce the argument here.

Theorem 7 (Weiss). $\mathcal{O} : \mathbf{M}_T \rightarrow \mathbf{Top}$ is an equivalence of categories.

Proof. In view of Theorem 5 we need only find an inverse to \mathcal{O} relative to objects. Take any topological space (X, τ) and construct an $\Omega(\tau)$ -space (X, d) with

$$d(x, y) = \{F \in [\tau]^{<\omega} \mid \text{for all } U \in F \text{ if } x \in U \text{ then } y \in U\}.$$

Let $x \in U \in \tau$, denote $\epsilon = \{\emptyset, \{U\}\}$ and notice

$$\begin{aligned} y \in B_{\epsilon}(x) &\Rightarrow d(x, y) \prec \epsilon \\ &\Rightarrow d(x, y) \supseteq \epsilon \\ &\Rightarrow y \in U. \end{aligned}$$

Thus, $\mathcal{O}[(\Omega(\tau), X, d)] = (X, \tau)$ and the equivalence is shown. \square

3. TOPOLOGICAL NOTIONS IN \mathbf{M}_T

We begin by illustrating some basic topological properties in \mathbf{M}_T . Given a continuity space (X, V, d) with $x \in X$ and $C \subseteq X$ we denote

$$d(x, C) = \bigwedge_{y \in C} d(x, y),$$

in line with point-to-set distances in metric spaces.

Lemma 8. *For a continuity space (X, V, d) with $x \in X$ and $C \subseteq X$:*

- (1) $d(x, C) > 0$ if, and only if, there exists $\epsilon \succ 0$ with $B_\epsilon(x) \cap C = \emptyset$,
- (2) the topological closure of C is simply $\{z \mid d(z, C) = 0\}$, and thus
- (3) C is closed if, and only if, $d(x, C) = 0 \implies x \in C$.

Proof. The latter two follow (1) and to prove this claim assume that $d(x, C) = 0$. This happens precisely when for all $\epsilon \succ 0$,

$$\epsilon \succ d(x, C) = \bigwedge_{y \in C} d(x, y).$$

Equivalently, for all $\epsilon \succ 0$ we can find $y \in C$ with $d(x, y) \prec \epsilon$. This completes the proof. \square

In what follows we adopt standard set-theoretic notation: given a pair of sets X, Y the symbol X^Y denotes the collection of all functions from Y to X and for a product $X = \prod_{i \in I} X_i$, standard projections will be denoted by $\pi_j(X)$ for $j \in I$. For (V, X, d) , $C \subseteq X$ and any $R \in V_{\prec}^X$ we denote

$$B_R(C) = \bigcup_{x \in C} B_{R(x)}(x).$$

The following are straightforward to verify.

Lemma 9. *Given a continuity space (V, X, d) , then for all $x, y \in X$ and closed $C \subseteq X$:*

- (a) $d(x, y) = d(y, x) = 0$ implies $x = y$ if, and only if, $\mathcal{O}[(V, X, d)]$ is Kolmogorov.
- (b) $d(x, y) = 0$ implies $x = y$ if, and only if, $\mathcal{O}[(V, X, d)]$ is Fréchet.
- (c) $\bigvee_{R \succ 0} d(x, B_R(C)) = 0$ implies $d(x, C) = 0$ if, and only if, $\mathcal{O}[(V, X, d)]$ is Regular

Denote \mathbf{M}_K , \mathbf{M}_F and \mathbf{M}_R the full subcategories of \mathbf{M}_T satisfying the conditions of part (a), (b) and (c) from the above lemma, respectively. As aforementioned, complete regularity also has a metric counterpart: symmetry. However, their equivalence will be neither straightforward as the ones above nor an ‘if, and only if,’ statement. Before we establish this equivalence we will focus momentarily on other basic topological constructions in \mathbf{M}_T .

3.1. Subspaces and Products. Observe that if given a topological space (X, τ) and a continuity space (V, X, d) with $\mathcal{O}[(V, X, d)] = (X, \tau)$, then for any $Y \subseteq X$ the subspace topology (Y, τ_Y) is equivalent to $\mathcal{O}[(V, Y, d_Y)]$ where d_Y is the restriction of d to Y . As it will soon become apparent, all other (co)limits will require of largely more delicate constructions. Indeed, consider for example the case of finding the product of two metric spaces: (\mathbb{R}, X, d) and (\mathbb{R}, Y, m) . It is well-known what this product looks like. In particular, we know that it can be of the form $(\mathbb{R}, X \times Y, s)$, where $s : (X \times Y)^2 \rightarrow \mathbb{R}$ is usually given as some combination of d and m . When faced with a pair of arbitrary continuity spaces $(V, X, d), (W, Y, m)$ we can only be sure that the underlying set must be isomorphic to $X \times Y$. The value quantale for this product is an entirely different matter. An initial guess could be $V \times W$ ordered pointwise. This, in general, will not work: the well-above elements, in all but the most degenerate of cases, do not form a filter. The reader can verify this simply by computing $\mathbb{R} \times \mathbb{R}$. As we shall shortly illustrate, and perhaps not surprisingly, a suitable value quantale for the above scenario can be developed from the original suggestion. In the what follows, we refer to finite sequences of points in a space as **walks**. We will denote such walks as either x_1, \dots, x_n or simply $w(x_1, x_n)$, $v(x_1, x_n)$, $u(x_1, x_n)$, etc, where the starting and ending point are x_1 and x_n , respectively.

Definition 10. *Given a continuity space (V, X, d) , a walk $w(x_1, x_n)$ in X and $R \subseteq V_{\prec} \times X$ we say that R **admits** $w(x_1, x_n)$, and write $R \vdash (x_1, x_n)$, provided that for every $i \leq n - 1$ we can find $(\epsilon, x_i) \in R$ with $x_{i+1} \in B_{\epsilon}(x_i)$ for all $i \leq n - 1$.*

Notice that this new relation admits *concatenation of walks*: $R \vdash w(x_1, x_n)$ and $R \vdash u(y_1, y_m) \Rightarrow R \vdash v(x_1, y_m) = x_1, \dots, x_n, y_1, \dots, y_m$. Consider a family $\{(V_i, X_i, d_i)\}_{i \in I}$ of continuity spaces, and put $X = \prod X_i$ and $U = [\prod_0 (V_i)_{\prec}]^X$, where $\prod_0 (V_i)_{\prec}$ denotes the collection of all tuples of length I with all but a finite number of coordinates being \top_i . As it stands U is not a value quantale, however, all of the information needed for forging the appropriate one lies within it. Put $V = \Omega(U)$ - a value quantale - and for a pair $a, b \in X$ let

$$a \triangle b = \{A \in [U]^{<\omega} \mid \forall R \in A, \exists w(a, b) \text{ with } \pi_i(R) \vdash \pi_i[w(a, b)], \forall i \in I\}.$$

Obviously, $\pi_i(R) \vdash \pi_i[w(a, b)]$ means that $\pi_i(R)$ admits the projection of the walk $w(a, b)$ onto the i^{th} coordinate. Notice that $a \triangle b$ is closed downwards and thus $a \triangle b \in V$. We then define $d : X^2 \rightarrow V$ as $d(a, b) = a \triangle b$ and note that $d(x, x) = \perp_V$ for all $x \in X$. Also, given a triplet $x, y, z \in X$:

$$\begin{aligned} d(x, y) + d(y, z) &= d(x, y) \cap d(y, z) \\ &= x \triangle y \cap y \triangle z \\ &\subseteq x \triangle z && \text{(by concatenation of paths)} \\ &= d(x, z). \end{aligned}$$

Consequently, $d(x, y) + d(y, z) \geq d(x, z)$ and (V, X, d) is a continuity space.

Theorem 11. *For a family $\{(V_i, X_i, d_i)\}_{i \in I}$ of continuity spaces and (V, X, d) as defined above we have that*

$$(V, X, d) = \prod_{i \in I} (V_i, X_i, d_i).$$

That is, $\mathcal{O}[(V, X, d)]$ is the product of the family $\{\mathcal{O}[(V_i, X_i, d_i)]\}_{i \in I}$.

Proof. First we show that $\mathcal{O}[(V, X, d)]$ is as least as fine as the product space. Fix a $j \in I$, an $x \in X_j$, and an $\epsilon \in V_j$, and put $B = \pi_j^{-1}(B_\epsilon(x))$. Choose $f_\epsilon \in [\prod_0(V_i)_\prec]^X$ so that when given a $z \in B$, $\pi_j(f_\epsilon(z)) = \delta$ with $B_\delta(\pi_j(z)) \subseteq B_\epsilon(x)$ and when $i \neq j$, $\pi_i(f_\epsilon(z)) = \top_i$. For all $z \notin B$ the choice is irrelevant but we make one nonetheless: let $\pi_i(f_\epsilon(z)) = \top_i$. Lastly, fix any $p \in X$ with $\pi_j(p) = x$ and observe that $B_{\{f_\epsilon\}}(p) = B$.

To prove the reverse containment, begin by picking any $\epsilon \succ \perp_V$ and fixing any $p \in X$. Notice that since $|\epsilon|$ is finite, the point-wise meet of all functions in $\cup \epsilon$ - a finite number of them - defines a function in $[\prod_0(V_i)_\prec]^X$: denote it by f . Let $F \subseteq I$ represent the finite set of indices where $\pi_i(f) \neq \top_i$; indeed, F can be empty. In any case, the basic open set

$$\bigcap_{j \in F} \pi_j^{-1} [B_{\pi_j(f)}(\pi_j(p))]$$

in the product topology is contained in $B_\epsilon(p)$. Thus, the product topology is at least as fine as the one generated by (V, X, d) and the proof is complete. \square

Remark 12. *The metric construction of the box topology is identical to the one outlined above with the slight difference that $U = [\prod(V_i)_\prec]^X$. It is also worth noticing that the method of walks, which we recycle in the sequel when dealing with colimits, could have been employed when describing subspaces, thus illustrating a general theme among all topological (co)limits.*

Another striking motif among these constructions - already partially witnessed - is the similarity between the underlying sets and their value quantale counterpart. This observation will be reinforced in what follows.

3.2. Sums and Quotients. Consider a family $\{(V_i, X_i, d_i)\}_{i \in I}$ of continuity spaces, and put $X = \coprod X_i$ and $U = [\coprod (V_i)_{\prec}]^X$, where \coprod denotes the usual disjoint union. Turn U into a value quantale by letting $V = \Omega(U)$ and, again, for a pair $a, b \in X$ let

$$a \triangle b = \{A \in [U]^{<\omega} \mid \forall R \in A, \exists w(a, b) \text{ with } R \vdash w(a, b)\}.$$

Notice that in the above definition no finite walk in X with elements from different factors of X is admitted by any $R \in U$. That is, for $a \in X_k$ and $b \in X_j$ with $j \neq k$ we have $a \triangle b = \emptyset = \top_V$. Consequently, for all $a, b \in X$ we have $a \triangle b \in V$ and we, thus, define $d : X^2 \rightarrow V$ by $d(a, b) = a \triangle b$. Also, $d(a, a) = \perp_V$ for all $a \in X$ and that concatenation of paths yields the triangle inequality. By design, for a fixed $i \in I$ the topology generated by the restriction of d onto X_i matches exactly the one generated by (V_i, X_i, d_i) and, as one would expect, points from different factors of X are as far away from each other as possible. Thus, the V -space (X, d) clearly generates the topological sum on X .

Theorem 13. *For a family $\{(V_i, X_i, d_i)\}_{i \in I}$ of continuity spaces and (V, X, d) as defined above we have that*

$$(V, X, d) = \coprod_{i \in I} (V_i, X_i, d_i).$$

We complete this section by illustrating quotients; the reader versed with the previous constructions should find the following rather familiar. Begin with a continuity space (Y, W, m) and an equivalence relation $\sim \in Eq(Y)$. Denote $X = Y/\sim$. We seek a value quantale V and a distance assignment $d : X^2 \rightarrow V$ that will generate that quotient topology on X . As with equalisers, the value quantale V will be based on $U = (V_{\prec})^X$. Put $V = \Omega(U)$ and let $a \triangle b \in V$ be the collection of $A \in [U]^{<\omega}$ so that for any $R \in A$ there exists a walk $a = x_1, \dots, b = x_n \in X$ with $x_i \sim x_{i+1}$ provided i is even and $x_{i+1} \in B_{R(x_i)}(x_i)$ when i is odd. As expected, for a pair $a \sim b$ we have $a \triangle b = \perp_V$ and, yet again, concatenation of paths yields the triangle inequality. Finally, letting $d(a, b) = a \triangle b$ one can readily verify that (V, X, d) is a continuity space. Let $q : Y \rightarrow X$ be the quotient function giving rise to \sim , choose any $x \in X$ and $\epsilon \succ \top_V$. Put

$$\delta = \bigwedge_{R \in \cup \epsilon} R(x)$$

and notice that $\delta \in W$ since $\cup \epsilon$ is finite. By design, $q[B_\delta^m(x)] \subseteq B_\epsilon^d(q(x))$ and the latter epsilon ball is a saturated open set. Consequently, the V -space (X, d) generates the quotient topology on X .

Theorem 14. *For a continuity space (W, Y, m) and $\sim \in Eq(Y)$ the continuity space (V, X, d) as defined above generates the quotient space.*

Remark 15. *Notice that when $\sim = \emptyset$ the previous theorem states that $\mathcal{O}(W, Y, m) = \mathcal{O}(V, X, d)$. That is, defining open sets by means of walks is equivalent to the standard use of epsilon balls. This rather obvious fact is really what inspired the constructions for the above illustrated (co)limits.*

4. SYMMETRY AND COMPLETE REGULARITY

Characterizing complete regularity purely in metric terms is considerably more involved than the previous separations axioms. As a matter of fact, this latter axiom does not allow for “if, and only if,” statements but rather equivalences between categories via different functors than M and \mathcal{O} . This is far from an unexpected result: symmetry in Kopperman’s value semigroups is also equivalent to complete regularity. Moreover, uniformities yield only completely regular spaces and any completely regular topology has a corresponding uniform space. Dyadics are frequently employed when showing complete regularity. What we develop next are suitable embeddings of such within value quantales. The proof of the following can be found in [5] pg. 264.

Lemma 16 (Flagg). *For a value quantale $(V, \leq, +)$ if $\epsilon \succ 0$ then there exists $\delta \succ 0$ so that $\epsilon \succ 2\delta$.*

This is key to show complete regularity. Put $\mathcal{D} = \{\frac{i}{2^j} \leq 1 \mid i, j \in \mathbb{N}\}$. In a nutshell, given any $\epsilon \succ 0_V$ in a value quantale V , the previous lemma is enough to guarantee an order-preserving function $f : \mathcal{D} \rightarrow \{\delta \in V \mid \delta \leq \epsilon\}$. In turn, as we shall see, it is possible to generate a nested collection of open sets (indexed by \mathcal{D}) about any point $x \in X$. Take any $\epsilon \succ 0_V$. We are guaranteed at least one $\delta_1 \succ 0_V$ so that $2\delta_1 \prec \epsilon$. Similarly, since $\delta_1 \succ 0_V$ we can find $\delta_2 \succ 0_V$ so that $2\delta_2 \prec \delta_1$ and so on. Thus, we have $\{\delta_i \mid i \in \mathbb{N}\}$ so that $2\delta_{i+1} \prec \delta_i$. Next we define, $\frac{\epsilon}{2^i} := \delta_i$ (where we let $\delta_0 := \epsilon$). Clearly, $2\frac{\epsilon}{2} \leq \epsilon$ and equality is not at all guaranteed: take the lattice $\{\perp, \top\}$ as an example. At this stage all powers of $\frac{1}{2}$ have been suitably defined. Recall that any dyadic number can be expressed uniquely as a sum of products of $\frac{1}{2}$; given any $n \in \mathcal{D}$ we can express $n = \sum_{i=0}^{\infty} f(i)_n \frac{1}{2^i}$ where $f(i)_n = 0$ or 1 (in a sense this function defines n). In general we let, for $n \neq \frac{1}{2^i}$ (since we’ve already defined those) $n\epsilon := \sum_{i=0}^{\infty} f_n(i)\epsilon$ and it is simple to show that for $n \leq m \in \mathcal{D}$ then $n\epsilon \leq m\epsilon$. The following proof is inspired on a result by Kopperman ([6] pg. 97) and we recycle some of his notation here.

Lemma 17. *Any symmetric continuity space yields a completely regular topology.*

Proof. Take any symmetric continuity space (V, X, d) , with $V = (V, \leq_V, +_V)$ and fix an $x_0 \in O \in \tau_V$ (i.e. the topology generated by (V, X, d)) and notice that there exists an $\epsilon \succ 0$ so that $\overline{B_\epsilon(x)} \subseteq O$ (that is, there is a closed set of radius ϵ about x_0 entirely contained within O). Indeed, since $x_0 \in O$ then there exists a $p \succ 0$ for which $B_p(x_0) \subseteq O$. Since $p \succ 0$ we can find $\epsilon \succ 0$ for which $2\epsilon \prec p$. In turn, $d(x_0, y) \leq \epsilon \Rightarrow d(x_0, y) \prec p$ and $y \in B_p(x_0)$. For this ϵ we can, as shown above, generate at least one order-preserved copy of \mathcal{D} within $\{\delta \succ 0 \mid \delta \prec \epsilon\}$. Let us fix one such copy and continue with the proof. Let $M_\epsilon : V \rightarrow [0, 1]$ by

$$M_\epsilon(a) = \begin{cases} 1 & \text{if } \{n \in \mathcal{D} \mid a \leq n\epsilon\} = \emptyset, \\ \inf\{n \in \mathcal{D} \mid a \leq n\epsilon\} & \text{otherwise.} \end{cases}$$

and observe that M_ϵ preserves the triangle inequality relative to $[0, 1]$. Indeed, if $c \leq_V a +_V b$, and $a \leq_V r\epsilon$ and $b \leq_V s\epsilon$ for a pair $r, s \in \mathcal{D}$ then $c \leq_V a +_V b \leq_V r\epsilon +_V s\epsilon \leq_V (r + s)\epsilon$. Hence, $r + s \in \{n \in \mathcal{D} \mid c \leq n\epsilon\}$ and

$$M_\epsilon(c) \leq \inf\{n + m \mid a \leq n\epsilon \text{ and } b \leq m\epsilon\} = M_\epsilon(a) + M_\epsilon(b).$$

Next we define the auxiliary function $g : X \rightarrow [0, 1]$ by

$$g(y) = \min\{M_\epsilon(d(x_0, y)), 1\}$$

and proceed to that show g is continuous; the actual point-closed separating function is defined subsequently. First notice that for any $y, z \in X$, since $d(x_0, z) \leq d(x_0, y) + d(y, z)$ we have $M_\epsilon(d(x_0, z)) \leq M_\epsilon(d(x_0, y)) + M_\epsilon(d(y, z))$ and, consequently, $g(z) \leq g(y) + M_\epsilon(d(y, z))$. It follows that $g(z) - g(y) \leq M_\epsilon(d(y, z))$ and, by symmetry of d , that $|g(y) - g(z)| \leq M_\epsilon(d(y, z))$.

For continuity of g we require that for any $x \in X$ and any $p > 0$ we can find $\delta \succ 0_V$ so that $d(x, y) \prec \delta \Rightarrow |g(x) - g(y)| < p$. Choose any $p > 0$ and take any $n \in \mathcal{D}$ so that $n < p$. Notice that if $d(y, z) \prec n\epsilon$ then $|g(y) - g(z)| \leq M_\epsilon(d(y, z)) \leq n < p$ and thus g is continuous. Lastly, define $f : X \rightarrow [0, 1]$ as $f(x) = \max\{0, 1 - g(x)\}$. Hence, f is continuous with $x \mapsto 1$ and $y \mapsto 0$ for any $y \notin \overline{B_\epsilon(x)}$. \square

A completely regular space coincides with initial topology from its collection of all continuous real-valued functions. Equivalently, it is also the smallest one making all of its point-closed set separating functions continuous. Indeed, let τ be any completely regular topology on a set X and denote τ^* to be the one generated from all of the continuous point-closed set separating functions one τ . Let $f : X \rightarrow \mathbb{R}$ be any τ -continuous function and fix any $(\alpha, \beta) \subseteq \mathbb{R}$. Denote $(\alpha, \beta)_f = f^{-1}(\alpha, \beta)$ and notice that for any $x \in (\alpha, \beta)_f$ we can find a τ -continuous (and thus τ^* -continuous) point-closed set separating function $g_x : X \rightarrow [0, 1]$

for which $g_x(x) = 0$ and $f(y) = 1$ for any $y \in X \setminus (\alpha, \beta)_f$. It is then simple to observe that

$$\bigcup_{x \in (\alpha, \beta)_f} g_x^{-1}[0, 1) = (\alpha, \beta)_f,$$

an open set in τ^* . Hence, f is also τ^* -continuous. The idea behind the following theorem is to construct a symmetric continuity space based on such a collection. The construction relies on making all such real-valued functions continuous and generating the original completely regular topology.

Lemma 18. *Any completely regular topology can be generated by a symmetric continuity space.*

Proof. Let (X, τ) be a completely regular topology and put F as the collection of all continuous $[0, 1]$ -valued functions on X that separate points from closed sets. Let $V = [0, 1]^F$ and

$$K = \{f \in (0, 1]^F \mid f(x) = 1 \text{ for all but finitely many } x \in F\}.$$

As it stands, neither V nor K are value quantales (the well-above 0 elements do not form a filter). What we do, by combining Kopperman's and Flagg's ideas, is to embed V into a suitable value quantale where the well-above 0 elements will be exactly the images of K . For a pair $f, g \in V$, we say that g is **way-above** f , $f \ll g$, if $g(x) > f(x)$ for all x for which $f(x) < 1$. A **round filter** (cf [5] pg. 275) $p \subseteq K$ is one for which:

- $\top \in p$,
- for all $f, g \in K$ if $f \in p$ and $f \ll g$ then $g \in p$, and
- for any $f \in p$, $\exists g \in p$ so that $g \gg f$.

Following Flagg's notation, we let $\Gamma(K)$ denote the collection of all round filters on K ; Flagg shows that $\Gamma(K)$ is indeed a value quantale when ordered by reverse inclusion and addition is taken to be intersection. For completion, we repeat the proof here.

CLAIM: $\Gamma(K)$ is a value quantale.

Proof. That $\Gamma(K)$ is a complete lattice follows immediately. For all $f \in K$ put $\hat{f} = \{g \in K \mid g \gg f\}$. If for some set $\{s_i\} \subseteq \Gamma(K)$, $\bigwedge \{s_i\} = \perp$ then $f \in s_j$ for some j , and for any $p \subseteq \hat{f}$ we have $p \subseteq s_j$. Conversely, one can take $\perp_{\Gamma(K)} = \bigwedge_{f \in K} \hat{f}$ to notice that only a $p \in \Gamma(K)$ so that $p \subseteq \hat{f}$ will be well-above $\perp_{\Gamma(K)}$. It is then easy to verify that $\Gamma(K)$ is completely distributive. Lastly, distributivity of unions over intersections makes $\Gamma(K)$ a value quantale. \square

Next, we embed V within $\Gamma(K)$ and generate the desired topology using $\Gamma(K)$. The embedding which we denote by Ψ , is the obvious

one: for any $f \in V$, $f \mapsto \hat{f}$ and notice that $\wedge \hat{f} = f$ so that Ψ is an embedding. In order to simplify the proof we define two functions: one will go from $X \times X$ into V and it will generate the other (the actual metric on X) from $X \times X$ into $\Gamma(K)$. The first function $m : X \times X \rightarrow V$ is done coordinate-wise. That is, for any pair $x, y \in X$ we have $m(x, y)(f) = |f(x) - f(y)|$, since m literally evaluates functions from V into $[0, 1]$. Obviously, the indiscrete topology is trivially completely regular. We treat this pathological case by letting all distances equal 0. The second one works as follows: $d : X \times X \rightarrow \Gamma(K)$ so that

$$(x, y) \mapsto \Psi(m(x, y)) = \widehat{m(x, y)}.$$

Next we show that $(\Gamma(K), X, d)$ generates (X, τ) . First we show that for any $p \in \Gamma(K)_{<}$, $B_p(x) \in \tau$ for any $x \in X$. Notice that

$$\begin{aligned} y \in B_p(x) &\Leftrightarrow d(x, y) \prec p \\ &\Leftrightarrow \widehat{\wedge p} \subseteq d(x, y) \\ &\Leftrightarrow \wedge p \gg m(x, y). \end{aligned}$$

By design, $\wedge p \gg m(x, y)$ if, and only if, for all $h_i \in F$ ($1 \leq i \leq n$) so that $p_i := \wedge p(h_i) < 1$, by definition, $p_i > m(x, y)(h_i) = |h_i(x) - h_i(y)|$. In turn we get $y \in B_p(x)$ if, and only if, $h_i(x) - p_i < h_i(y) < h_i(x) + p_i$ and the latter happens precisely when $y \in \bigcap_1^n h_i^{-1}[(h_i(x) - p_i, h_i(x) + p_i)]$. Thus, $B_p(x)$ is indeed open in τ .

Lastly, take $x \in O \in \tau$ and any $f \in F$ so that $f(x) = 1$ and $f(y) = 0$ for all $y \notin O$. Here is where it becomes clear that we need only look at continuous point-closed set separating functions from τ . Let $h \in K$ so that $1 > h(f) > 0$ and $h(g) = 1$ for all $g \neq f$. If $y \in B_{\hat{h}}(x)$ then $d(x, y) \prec \hat{h}$ yielding that $\hat{h} \subseteq d(x, y)$. Consequently, for $m(x, y)(g) < 1$ we get $h(g) > m(x, y)(g) \Rightarrow h(f) > m(x, y)(f) = |f(x) - f(y)|$. By design, $h(f) < 1$ and thus $f(y) > 0$. Hence, $y \in O$ and $B_{\hat{h}}(x) \subseteq O$. \square

Corollary 19. *The full category \mathbf{M}_S of symmetric continuity spaces is equivalent to the category of \mathbf{CReg} of all completely regular topologies.*

The following follows immediately from the previous results.

Corollary 20. *The categories \mathbf{M}_K , \mathbf{M}_F , \mathbf{M}_R , and \mathbf{M}_S are reflective subcategories of \mathbf{M}_T .*

There exist at least three obvious ways to *symmetrise* a continuity space (and thus ‘completely regularise’ a topology): given a topology τ , which is not completely regular, we can generate its corresponding continuity space, say (V, X, d) . Since d is not symmetric (for then τ would be completely regular) we can construct its dual $d^* : X \times X \rightarrow V$ so that $d(x, y) = d^*(y, x)$ for all $x, y \in X$. Let τ^* be the topology generated by (V, X, d^*) . We then have three obvious candidates for a *symmetrisation* of (V, X, d) :

- (a) (V, X, d_\vee) so that $m_\vee(x, y) = d(x, y) \vee d^*(x, y)$,
- (b) (V, X, d_+) so that $m_+(x, y) = d(x, y) + d^*(x, y)$ and
- (c) (V, X, d_\wedge) so that

$$d_\wedge(x, y) = \bigwedge_{\gamma} \left(\sum_i^n d(a_i, a_{i+1}) \wedge d^*(a_i, a_{i+1}) \right)$$

where the sums are indexed over all walks γ of the form $x = a_1, \dots, y = a_n$.

Flagg proves that (b) generates $\tau \vee \tau^*$. The same is true for (a): clearly, $\mathcal{O}(V, X, d_\vee) \geq \tau \vee \tau^*$ and since $\forall p \succ 0$, $B_p^m(x) = B_p^d(x) \cap B_p^{d^*}(x)$ then $\mathcal{O}(V, X, d_\vee) \leq \tau \vee \tau^*$. Thus, $\mathcal{O}(V, X, d_\vee) = \mathcal{O}(V, X, d_+)$. Notice that in (c), $\mathcal{O}(V, X, d_\wedge) \leq \tau \wedge \tau^*$ and equality is not at all guaranteed; see [3] Theorem 4.2 (b).

None of these symmetrisations is equivalent to the left adjoint of $\mathbf{CReg} \hookrightarrow \mathbf{Top}$. Since (a) and (b) generate topologies finer than the original one the claim is then clear for such cases. In contrast, (c) is not as straight forward. Let $X = \{x\} \cup \{x_n\}_{n \in \mathbb{N}}$ and $d : X \times X \rightarrow \mathbb{R}$ so that:

$$d(y, z) = \begin{cases} \frac{1}{n} & \text{if } y = x_n \text{ and } z = x \\ 1 & \text{otherwise.} \end{cases}$$

The functor \mathcal{O} maps (V, X, d) to the discrete topology on X (a completely regular topology) but the symmetrisation of (X, d) via (c) is mapped by \mathcal{O} to a strictly coarser topology than the discrete topology on X . Indeed, the sequence (x_n) converges to x in (V, X, d_\wedge) , thus (c) can not even define a left adjoint to the inclusion $\mathbf{M}_S \hookrightarrow \mathbf{M}_T$.

As a final observation, notice that $\mathcal{O}[(V, X, d^*)] = \mathcal{O}[(V, X, d_\wedge)]$ and that $\mathcal{O}[(V, X, d)] = \mathcal{O}[(V, X, d_\vee)]$, thus neither (a) nor (b) give rise to a right adjoint to the inclusion functor $\mathbf{M}_S \hookrightarrow \mathbf{M}_T$.

Question 1. *What are the left and right adjoint of $\mathbf{M}_S \hookrightarrow \mathbf{M}_T$?*

Question 2. *What is the equivalent to the left adjoint of $\mathbf{CReg} \hookrightarrow \mathbf{Top}$?*

5. CONCLUSION

We conclude this manuscript by reestablishing some classical results through this metric formalism and highlighting their almost immediate evidence. The metric versions of separation axioms T_0 , T_1 and T_3 are easily shown to be productive. As illustrated in Section 3.1 it is clear from the construction of products in \mathbf{M}_T that a given product is symmetric precisely when all of its factors are also. Thus, in conjunction with the equivalence $\mathbf{M}_T \simeq \mathbf{Top}$ we can easily establish the following.

Theorem 21. *Any product of continuity spaces is symmetric precisely when all of its factors are also. Consequently, the same is true for Tychonoff spaces.*

Affine to our work is a recent publication by Weiss (see [9]) establishing a metric, and *positive*, characterisation of connectedness.

Theorem 22 (Weiss). *A continuity space (V, X, d) generates a connected topological space if, and only if, for all $a, b \in X$ and for all $R \in V_{\prec}$ there exists a walk $a = x_1, \dots, b = x_n$ with either $d(x_i, x_{i+1}) \prec R(x_i)$ or $d(x_{i+1}, x_i) \prec R(x_{i+1})$.*

Corollary 23. *Any product of continuity spaces is connected precisely when all of its factors are as well.*

Proof. By construction of the well-above relation in the product space, the failure of the product space (resp. any one of the factor spaces) to satisfy connectedness implies the failure of at least one its factor spaces (resp. the product space) also. □

Remark 24. *The reader should notice that, by design, proving that quotients preserve connectivity in this metric setting is also straight forward.*

The scope for future research within this setting is overarching rather than specific. For instance, consider the notion of compactness; a hugely important topological property that has been extensively studied by topologists and logicians alike for nearly two centuries. There exists a large plethora of equivalent definitions for compactness (open covers, ultrafilters, nets, complete accumulation points, etc) and, at first glance, it might not be clear what its "true metric equivalent" is. We investigate the metric equivalent of compactness and many other affine properties (paracompactness, pseudo-compactness, etc) in [2].

ACKNOWLEDGMENTS

We would like to extend our sincere gratitude to Paul Szeptycki for his many helpful suggestions, in particular with the metric construction of (co)limits.

REFERENCES

- [1] Nathanael Leedom Ackerman. A fixed point theorem for contracting maps of symmetric continuity spaces. *Topology Proc.*, 47:89–100, 2016.
- [2] J. Bruno. Compactness in continuity spaces. *In preparation*.
- [3] J. Bruno and Ittay Weiss. Metric axioms: a structural study. *Topology Proc.*, 47:59–79, 2016.
- [4] Alveen Chand and Ittay Weiss. Completion of continuity spaces with uniformly vanishing asymmetry. *Topology Appl.*, 183:130–140, 2015.

- [5] R. C. Flagg. Quantales and continuity spaces. *Algebra Universalis*, 37(3):257–276, 1997.
- [6] Ralph Kopperman. All topologies come from generalized metrics. *Amer. Math. Monthly*, 95(2):89–97, 1988.
- [7] George Neal Raney. *Completely Distributive Complete Lattices*. 1953. Thesis (Ph.D.)—Columbia University.
- [8] Ittay Weiss. A note on the metrizability of spaces. *Algebra universalis*, 73(2):179–182, 2015.
- [9] Ittay Weiss. Metric characterisation of connectedness for topological spaces. *Topology Appl.*, 204:204–216, 2016.
- [10] Ittay Weiss. Metric constructions of topological invariants. *Topology Proc.*, 49:85–104, 2017.